Geometric Approach to Quantum Statistical Mechanics and Minimal Area Principle *

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Abstract

A geometric approach to some quantum statistical systems (including the harmonic oscillator) is presented. We regard the (N+1)-dimensional Euclidean *coordinate* system (X^i,τ) as the quantum statistical system of N quantum (statistical) variables (X^i) and one *Euclidean time* variable (τ). Introducing a path (line or hypersurface) in this space (X^i,τ), we adopt the path-integral method to quantize the mechanical system. This is a new view of (statistical) quantization of the *mechanical* system. It is inspired by the *extra dimensional model*, appearing in the unified theory of forces including gravity, using the bulk-boundary configuration. The system Hamiltonian appears as the *area*. We show quantization is realized by the *minimal area principle* in the present geometric approach. When we take a *line* as the path, the path-integral expressions of the free energy are shown to be the ordinary ones (such as N harmonic oscillators) or their simple variation. When we take a *hyper-surface* as the path, the system Hamiltonian is given by the *area* of the *hyper-surface* which is defined as a *closed-string configuration* in the bulk space. In this case, the system becomes a O(N) non-linear model. The two choices, (1) the *line element* in the bulk (X^i, τ) and (2) the Hamiltonian(defined as the damping functional in the path-integral) specify the system dynamics. After explaining this new approach, we apply it to a topic in the 5 dimensional quantum gravity. We present a *new standpoint* about the quantum gravity: (a) The metric (gravitational) field is treated as the background (fixed) one; (b) The space-time coordinates are not merely position-labels but are quantum (statistical) variables by themselves. We show the recently-proposed 5 dimensional Casimir energy (arXiv:0801.3064,0812.1263) is valid.

Keywords: harmonic oscillator, geometric view, minimal area principle, extra-dimensional model, path-integral, Casimir energy, O(N) non-linear sigma model, induced geometry, quantum gravity, uncertainty relation, space-time coordinates, hyper-surface

*The content was presented (talk in ICSF2010) in [1].

1 Introduction

In the quest for the fundamental structure of the space, time, and matter, the most advanced theories are the string theory, D-brane theory and M-theory[2]. They are beyond the quantum field theory in that the extended (in space) objects are treated as fundamental elements. Since the finding of AdS/CFT correspondence[3, 4, 5], various new ideas and techniques, developed for them, are imported into the *non-perturbative* analysis of the quantum field theories. In particular, the application to the material physics is marvelous: the heavy ion collision physics and the viscosity in the quark-gluon plasma([6, 7, 8] for review), superconductivity and superfluidity [9, 10, 11, 12], baryon mass spectrum in QCD[13, 14]. In this circumstance, two *new standpoints* about the space-time quantization appear. One is proposed by Hořava[15, 16]. He introduced Lifshitz's higher-derivative scalar theory and its renormalization group behavior into his idea about the new quantum gravity. Another one is revively given by E. Verlinde[17]. He emphasizes the entropic force (rather than the energetic force) and the thermodynamical behavior near the horizon (Hawking radiation). With this recent trend of the geometrical view, the statistical(thermal) view and the visco-elastic view , we present a new formalism where the quantum statistical system is treated purely in the geometrical way.

Let us mention the present situation of the space-time quantization (quantum gravity), because it is the concrete motivation of the present work. The space-time geometry is specified by the metric tensor field $g_{\mu\nu}(x)$ which appears in the definition of the line element $(ds^2)_{4D} = g_{\mu\nu}(x)dx^\mu dx^\nu$ $(\mu, \nu = 0, 1, 2, 3)$. One of most important problems of the present theoretical physics is the clarification of the *quantum role* of the metric (gravitational) field $g_{\mu\nu}$. We already have a long (nearly half century) history of the quantum gravity since Feynman[18] and DeWitt [19] pioneered. About one decade ago, inspired by the development of the string theory and the D-brane theory, a fascinating model of unification of forces was proposed. It is a 5 dimensional(dim) model with AdS_5 geometry and is called "Randall-Sundrum model" or the "warped model"[20]. This is a representative of the extra dimensional models. The most important purpose of the present work is to make this 5 dim model *legitimate* as the *quantum field theory*.

The AdS_5 space-time geometry is described as

$$\text{Warped Metric (y-expression)} \quad ds^2 = \mathrm{e}^{-2\omega|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \ -l \le y \le l \ , \tag{1}$$

where $\{\mu, \nu = 0, 1, 2, 3\}$, $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$. y is the extra coordinate. The parameter ω is the 5 dim (bulk) scalar curvature. l is the size parameter of the extra coordinate. We respect the periodicity: $y \to y + 2l$, and Z_2-parity: $y \leftrightarrow -y$. Instead of y, another

coordinate z is also used. [1]

$$\text{Warped Metric (z-expression)} \quad ds^2 = \frac{1}{\omega^2 z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = G_{MN}dX^M dX^N \ ,$$
$$|z| = \frac{1}{\omega}\mathrm{e}^{\omega|y|} \quad , \quad \frac{1}{\omega} < |z| < \frac{1}{T} \quad , \quad T \equiv \omega \mathrm{e}^{-\omega l} \quad ,$$
$$R_{MN} = 4\omega^2 G_{MN} \quad , \quad R = 20\omega^2 > 0 \quad , \quad \sqrt{-G} = \sqrt{-\det G_{MN}} = \frac{1}{(\omega|z|)^5} \quad , \qquad (2)$$

where $(X^M) \equiv (x^\mu, z)$, $\{M, N = 0, 1, 2, 3, 5\}$. [2] The flat (5D Minkowski) limit is obtained by $\omega \to 0$ in the y-expression (1).

$$\text{Flat Metric} \quad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad , \quad (X^M) = (x^\mu, y) \quad , \quad -l \leq y \leq l \quad , \qquad (3)$$

Traditional calculation[21, 22, 23] gives the Λ^5-divergent result for Casimir energy, on the above geometries, of 5D models. In the calculation, Casimir energy is expressed as the 5D space-momentum integral ($\int d^4p_E dy$ or $\int d^4p_E dz$) of some energy (density) function $F(\tilde{p}, y)$ or $F(\tilde{p}, z)$. (See Sec.5 for detail.) In ref.[22, 23], we claim the Λ^5-divergence comes from this 'naive' integration measure and should be replaced by some proper measure, based on close numerical calculation using some trial integration measures. Finally, Caasimir energy of the free fields (electromagnetism, free scalar theory) is *proposed* to be replaced by the following *path-integral*.

For Flat Geometry

$$-\mathcal{E}_{Cas}(l, \Lambda) = \int_{1/\Lambda}^{l} d\rho \int_{r(0)=r(l)=\rho} \prod_{a,y} \mathcal{D}x^a(y) \left\{ \int_0^l F_1\left(\frac{1}{r(\tilde{y})}, \tilde{y}\right) d\tilde{y} \right\}$$
$$\times \exp\left[-\frac{1}{2\alpha'} \int_0^l \sqrt{r'^2 + 1}\ r^3 dy \right] , \ r' = \frac{dr}{dy},$$

For Warped Geometry

$$-\mathcal{E}_{Cas}(\omega, T, \Lambda) = \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)=r(1/T)=\rho} \prod_{a,z} \mathcal{D}x^a(z) \left\{ \int_{1/\omega}^{1/T} F_2\left(\frac{1}{r(\tilde{z})}, \tilde{z}\right) d\tilde{z} \right\}$$
$$\times \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1}\ r^3 dz \right] \quad , \quad r' = \frac{dr}{dz} \quad , \qquad (4)$$

where $r = \sqrt{\sum_{a=1}^4 (x^a)^2}$. [3] ($\{x^a | a = 1, 2, 3, 4\}$ is the *Euclideanized* coordinates of $\{x^\mu | \ \mu = 0, 1, 2, 3\}$, $x^0 = ix^4$.) In the above proposal, the isotropy of the 4D world $\{x^a\}$ is assumed.

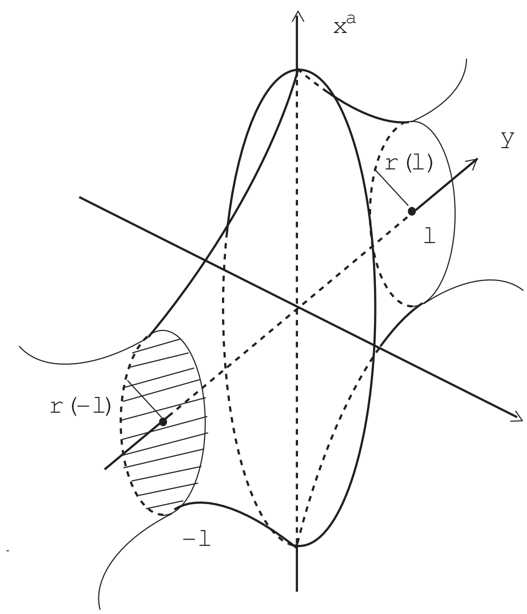
[1] z is defined by y as

$$z = \begin{cases} \frac{1}{\omega}\mathrm{e}^{\omega y} & y > 0 \\ 0 & y = 0 \\ -\frac{1}{\omega}\mathrm{e}^{-\omega y} & y < 0 \end{cases}$$

[2] T is *not* a temperature parameter but a IR parameter like l ($T = \omega \mathrm{e}^{-\omega l}$). The temperature appears later as β^{-1}. See eq.(7).

[3] The case $\alpha' \to \infty$ in (4) is essentially the traditional definition of Casimir energy.

Figure 1: N(=2) dim hypersurface in N+1 dim (Euclidean flat) space $(x^1, x^2, \cdots, x^N, y) = (x^a, y)$. Sphere S^{N-1}(circles in the figure) at y has the radius $r(y)$.



F_1 and F_2 are some energy density functions and will appear later in (49) and (51) respectively. Λ is the UV-cutoff parameter, $\mu \equiv \Lambda T/\omega$ is the IR-cutoff one and l is the periodicity(IR) one. The above path-integrals are over all paths of 4 dim *hypersurfaces* defined by

$$\text{Flat Geometry:} \qquad \sqrt{\sum_{a=1}^{4}(x^a)^2} = r(y) \quad , \quad -l \leq y \leq l \quad ,$$

$$\text{Warped Geometry:} \qquad \sqrt{\sum_{a=1}^{4}(x^a)^2} = r(z) \quad , \quad \frac{1}{\omega} \leq |z| \leq \frac{1}{T} \quad . \tag{5}$$

The form of the function $r(y)$ or $r(z)$ specifies the path of the hypersurface. See Fig.1 for the case of the N+1 dim space. This is a *closed string* configurartion. The *area* (4D volume) plays the role of *Hamiltonian* of the quantum statistical system $\{x^a\}$. F_i comes from the *matter-field* quantization and plays a role of the *energy 'operator'* in the path-integral over the 4D hyper-surface $r(y)$ or $r(z)$. The *string (surface) tension* parameter $1/2\alpha'$ is introduced. Note that the proposal (4) is obtained by taking the *new standpoint* that the (bulk) metric field $G_{MN}(X)$ is *not* field-quantized and is treated as a background field. [4]

[4] We consider that the form of $G_{MN}(X)$ is given by the field equation of the 'effective' action which is obtained after the *field* quantization of all *matter* fields. It is a fixed (or background) field in the quantization process of the space-time. See also Sec.5.

Instead we regard the 4 dim coordinates x^a as the quantum (statistical) variables, and the extra one, y or z, as Euclidean time. The new point, compared with the 5D Casimir energy calculation so far[21], is the introduction of the 'minimal area' factor $\exp(-\frac{1}{2\alpha'}\mathrm{Area}) = \exp(-\frac{1}{2\alpha'}\int\sqrt{\det(g_{ab})}d^4x)$ where g_{ab} is the *induced* metric on the hyper-surface (5). $\alpha' \to \infty$ limit, in (4), goes to the traditional Casimir energy. We will show, in this paper, the above-type *path-integral* very naturally appears in many quantum-statistical systems when we view them *geometrically*. We will show the proposed quantities (4) are valid. This is the final aim of this paper.

The content is organized as follows. We start with the simple quantum statistical system of one harmonic oscillator in Sec.2. We see the geometric approach works well by regarding the *extra coordinate* as the *Euclidean time*. This approach is shown to give exactly the same result as the ordinary quantization. We generalize the harmonic oscillator potential (elastic system) to the general one in Sec.3. In Sec.4 the one variable system is generalized to the system of N variables. We analyze the quantum statistical system in the N+1 extra dimensional Euclidean *geometry*. As the path, we have two choices: line and hypersurface. The O(N) *nonliner* model naturally appears by taking the path of hypersurface which is the *closed-string* configuration of a special type (5). We stress that taking the area as Hamiltonian is one realization of the *minimal area principle*. In Sec.5, we explain the meaning of the new definition of Casimir energy (4) and present a new treatment of the quantum gravity. We conclude in Sec.6. In Appendix A, the content of Sec.4 (N variables elastic system) is generalized to the general system.

2 Quantum Statistical System of Harmonic Oscillator

2.1 'Dirac' Type

Let us consider 2 dim Euclidean space (X,τ) described by the following metric.

$$\begin{gathered} ds^2 = dX^2 + \omega^2 X^2 d\tau^2 = G_{AB}dX^A dX^B \quad , \\ (X^A) = (X^1, X^2) = (X,\tau) \quad , \quad (G_{AB}) = \mathrm{diag}(1,\omega^2 X^2) \quad , \\ R_{AB} = 0 \quad , \quad R = G^{AB}R_{AB} = 0 \quad , \end{gathered} \tag{6}$$

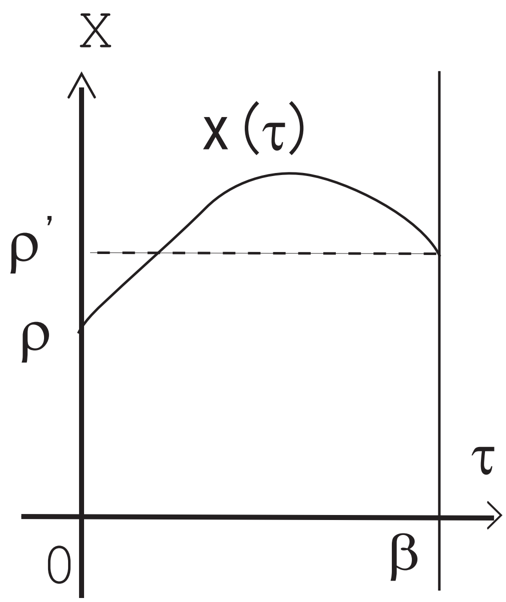
where $A,B = 1,2$. ω is the 'spring' constant with the dimension of mass. We impose the *periodicity* (period: β) in the direction of the extra dimension τ.

$$\tau \to \tau + \beta \quad . \tag{7}$$

This is a *way* to introduce the *temperature* $(1/\beta)$ in the system. Here we take a path $\{x(\tau),\ 0 \le \tau \le \beta\}$ in the 2D bulk space (X,τ) and the *induced* metric on the *line* is given by

$$\begin{gathered} X = x(\tau) \quad , \quad dX = \dot{x}d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \le \tau \le \beta \quad , \\ ds^2 = (\dot{x}^2 + \omega^2 x^2)d\tau^2 \quad . \end{gathered} \tag{8}$$

Figure 2: A path of line in 2D Euclidean space (X,τ). The path starts at x(0)=ρ and ends at x(β)=ρ'.



See Fig.2.

Then the *length* L of the path $x(\tau)$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + \omega^2 x^2} d\tau \quad . \tag{9}$$

We take the half of the length ($\frac{1}{2}L$) as the system Hamiltonian (*minimal length principle*). Then the free energy F of the system is given by [5]

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_\tau \mathcal{D}x(\tau) \exp\left[-\frac{1}{2} \int_0^\beta \sqrt{\dot{x}^2 + \omega^2 x^2} d\tau \right] \quad , \tag{10}$$

where the path-integral is done for all possible paths with the indicated boundary condition (b.c.). This quantum statistical system can be regarded as the 'fermionic partner' of the ordinary harmonic oscillator. [6]

[5] When we regard x as the space position (of a particle), the physical dimension is that of the length. Then the distribution function in (10) is, in general, $\exp(-L/2\alpha')$ where α'^{-1} is the tension parameter with the (length)$^{-1}$ dimension. We take $\alpha' = 1$ for simplicity. This note is valid for the following some models.

[6] The situation reminds us of the relation between Nambu-Goto action and Polyakov action in the string theory[24]. The introduction of an auxiliary variable helps to 'normalize' the square-root action (10). In this case, the geometric role of the auxiliary variable remains obscure.

2.2 Standard Type

Now we consider another type of 2 dim Euclidean space (X,τ) described by the following *line* element.

$$ds^2=\frac{1}{d\tau^2}(dX^2)^2+\omega^4X^4d\tau^2+2\omega^2X^2dX^2=\frac{1}{d\tau^2}(dX^2+\omega^2X^2d\tau^2)^2 \quad , \qquad (11)$$

where we put the following condition on the infinitesimal quantities, $d\tau^2$ and dX^2, in order to keep all terms of (11) in the same order.

$$\begin{aligned}&[\text{Line Element Regularity Condition}]\ :\\ &d\tau^2\sim O(\epsilon^2)\quad ,\quad dX^2\sim O(\epsilon^2)\quad ,\quad \frac{1}{d\tau^2}dX^2\sim O(1)\quad ,\end{aligned}\qquad (12)$$

where ϵ is an arbitrary infinitesimal parameter with the dimension of length. [7] Note that we do *not* have 2D metric in this case. (We *cannot* define the bulk metric $G_{AB}(X)$.) We impose the *periodicity* (period: β).

$$\tau\rightarrow\tau+\beta \quad . \qquad (13)$$

Here we take a path $\{x(\tau),\ 0\leq\tau\leq\beta\}$, and the *induced* metric on the line is given by

$$\begin{aligned}&X=x(\tau)\quad ,\quad dX=\dot{x}d\tau\quad ,\quad \dot{x}\equiv\frac{dx}{d\tau}\quad ,\quad 0\leq\tau\leq\beta\quad ,\\ &ds^2=(\dot{x}^2+\omega^2x^2)^2d\tau^2\quad .\end{aligned}\qquad (14)$$

In the bulk we do *not* have the metric, but on the path, we *do* have this *induced* metric. Then the *length* L of the path $x(\tau)$ is given by

$$L[x(\tau)]=\int ds=\int_0^\beta(\dot{x}^2+\omega^2x^2)d\tau \quad . \qquad (15)$$

Hence, taking $\frac{1}{2}L$ as the Hamiltonian (*minimal length principle*), the free energy F of the system is given by

$$\mathrm{e}^{-\beta F}=\int_{-\infty}^{\infty}d\rho\int_{\substack{x(0)=\rho\\ x(\beta)=\rho}}\prod_\tau\mathcal{D}x(\tau)\exp\left[-\frac{1}{2}\int_0^\beta(\dot{x}^2+\omega^2x^2)d\tau\right] \quad , \qquad (16)$$

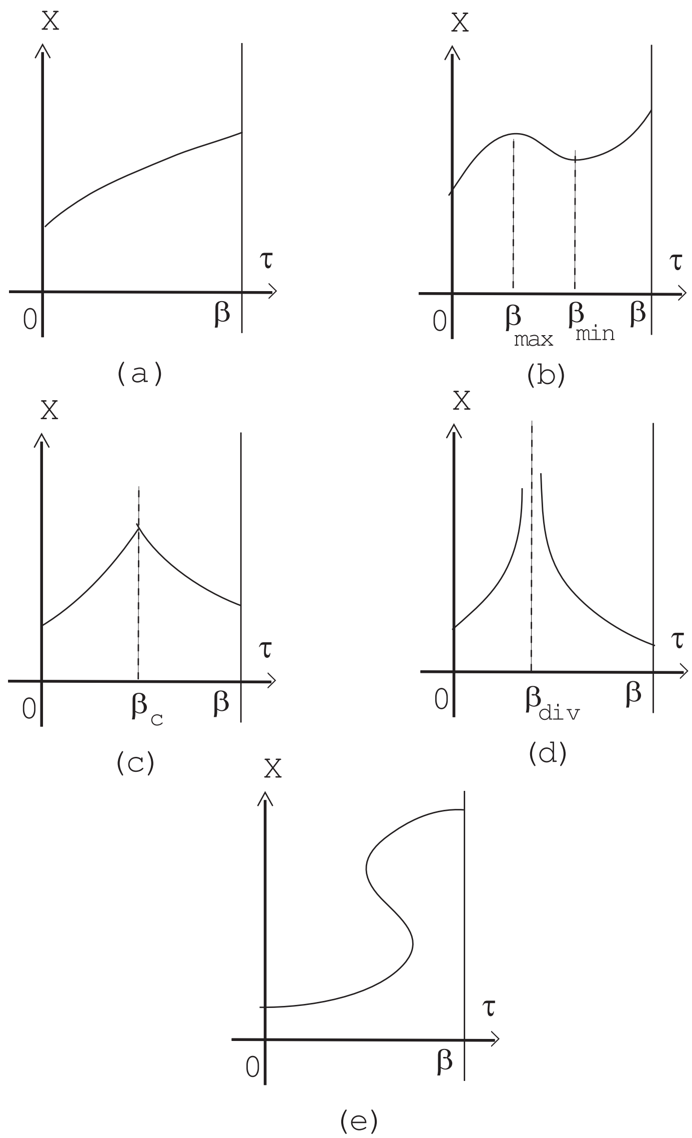
where the path-integral is done for all possible paths with the indicated b.c.. This is exactly the free energy of the harmonic oscillator. See Feynman's textbook[25]. [8]

Note that the condition (12) is necessary for the *elastic* view to the path.

[7] The condition (12) restricts the trajectory configuration (14) only to smooth-lines in the 2D bulk space, and excludes singular-lines which have some *singular* points (the derivative along τ can not be defined) between $0\leq\tau\leq\beta$. See Fig.3 for singular and regular lines. See the final section for the discussions about an interpretation of the condition (12).

[8] $F=\frac{\omega}{2}+\frac{1}{\beta}\ln(1-\mathrm{e}^{-\beta\omega}), E=<\frac{L}{2}>=\frac{\omega}{2}\coth(\frac{\omega\beta}{2})=\frac{\omega}{2}+\frac{\omega}{\mathrm{e}^{\omega\beta}-1}, S=\frac{1}{T}(E-F)=k\{\frac{\beta\omega}{2}\coth\frac{\beta\omega}{2}-\frac{\beta\omega}{2}-\ln(1-\mathrm{e}^{-\beta\omega})\}$

Figure 3: Singular and regular lines in 2D Euclidean space (X,τ). (a) regular line, simply increasing; (b) regular line, maximum at β_{max} and minimum at β_{min}; (c) singular line, different derivatives for $\beta \to \beta_c \pm 0$; (d) singular line, divergent at β_{div}; (e) singular line, multi-valued.



3 General Quantum Statistical System

We generalize the harmonic oscillator potential, $\frac{1}{2}\omega^2 X^2$, to the general one $V(X)$. As for $V(X)$, we have the following form in mind.

$$\frac{\omega^2}{2}X^2 + \frac{\lambda_3}{3!}X^3 + \frac{\lambda_4}{4!}X^4 + \cdots \quad , \tag{17}$$

where $\lambda_3, \lambda_4, \cdots$ are the coupling constants for additional terms.

3.1 'Dirac' Type

We start with the following metric in 2 dim Euclidean space (X, τ).

$$\begin{aligned}
ds^2 = dX^2 + 2V(X)d\tau^2 = G_{AB}dX^A dX^B \quad , \\
(X^A) = (X^1, X^2) = (X, \tau) \quad , \quad (G_{AB}) = \mathrm{diag}(1, 2V(X)) \quad , \\
(R_{AB}) = \begin{pmatrix} \frac{V''}{2V} - \frac{1}{4}\left(\frac{V'}{V}\right)^2, & 0 \\ 0 \quad , & V'' - \frac{1}{2}\frac{(V')^2}{V} \end{pmatrix} \quad , \quad R = G^{AB}R_{AB} = \frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2 \quad , \\
V' \equiv \frac{dV(X)}{dX} \quad , \quad V'' \equiv \frac{d^2V(X)}{dX^2} \quad ,
\end{aligned} \tag{18}$$

where $A, B = 1, 2$. Note that V(X) does *not* depend on τ. [9] We impose the periodicity (period: β) in the direction of the extra dimension τ (7). On a path $\{x(\tau),\ 0 \leq \tau \leq \beta\}$, the *induced* metric is given by

$$ds^2 = (\dot{x}^2 + 2V(x))d\tau^2 \quad , \quad 0 \leq \tau \leq \beta \quad . \tag{19}$$

Hence the *length* L of the path $x(\tau)$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + 2V(x)}d\tau \quad . \tag{20}$$

Taking the half of the length ($\frac{1}{2}L$) as the Hamiltonian, we get the free energy F as

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_\tau \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\dot{x}^2 + 2V(x)}d\tau\right] \quad . \tag{21}$$

[9] We furthermore note the new standpoint about the quantization of gravity (metric), in the present approach. The most familiar way is to regard the 'metric' $V(X)$ as a *field* variable at the point (X, τ) and quantize it field-theoretically. We do not take such way of quantization. We accept the potential form of $V(X)$ as a given one (background treatment) and do not treat $V(X)$ as the quantum (field) variable. Instead, we treat the coordinate X as the *quantum statistical* variable using the extra coordinate τ as the Euclidean time. See Sec.5 furthermore.

3.2 Standard Type

We start with the following line element.

$$ds^2 = \frac{1}{d\tau^2}(dX^2)^2 + 4V(X)^2 d\tau^2 + 4V(X)dX^2 = \frac{1}{d\tau^2}\left(dX^2 + 2V(X)d\tau^2\right)^2 \quad , \tag{22}$$

where we put the condition (12) on the infinitesimal quantities, $d\tau^2$ and dX^2, in order to keep all terms in the same order. The 2D bulk space do *not* have 2D metric. We impose the periodicity (period: β) (13). On a path $\{x(\tau),\ 0 \leq \tau \leq \beta\}$, we have the *induced* metric:

$$ds^2 = (\dot{x}^2 + 2V(x))^2 d\tau^2 \quad . \tag{23}$$

The *length* L is given by

$$L[x(\tau)] = \int ds = \int_0^\beta (\dot{x}^2 + 2V(x))d\tau \quad . \tag{24}$$

Taking $\frac{1}{2}L$ as the Hamiltonian, the free energy F is given by

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_\tau \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta (\dot{x}^2 + 2V(x))d\tau\right] \quad , \tag{25}$$

where the path-integral is done for all possible paths with the indicated b.c.. This is exactly the free energy of the quantum statistical system of one variable x in the general potential V(x).

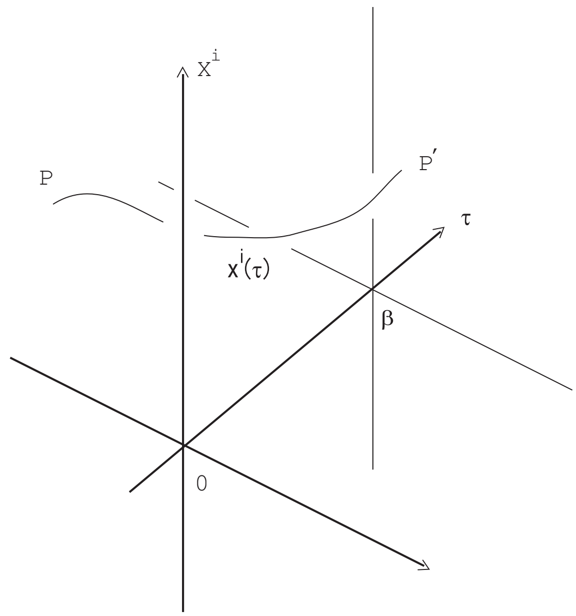
4 Quantum Statistical System of N Harmonic Oscillators and O(N) Nonlinear Model

4.1 'Dirac' Type of N Harmonic Oscillators and O(N) nonlinear system

Let us consider N+1 dim Euclidean space $(X^i, \tau), i = 1, 2, \cdots, N$ described by the following metric.

$$ds^2 = \sum_{i=1}^N (dX^i)^2 + \omega^2 d\tau^2 \sum_{i=1}^N (X^i)^2 = \sum_{i=1}^N (dX^i)^2 + 2V(r)d\tau^2 = G_{AB}dX^A dX^B \quad ,$$
$$A, B = 1, 2, \cdots, N, N+1; \quad X^{N+1} \equiv \tau \quad , \quad V(r) = \frac{1}{2}\omega^2 r^2 \quad ,$$
$$(G_{AB}) = \mathrm{diag}(1, 1, \cdots, 1, \omega^2 r^2) \quad , \quad r^2 \equiv \sum_{i=1}^N (X^i)^2 \quad . \tag{26}$$

Figure 4: A path of line $\{x^i(\tau)|i=1,2,\cdots,N\}$ in N(=2)+1 dim space. It starts at P=$(\rho_1,\rho_2,\cdots,\rho_N,0)$ and ends at P$'$=$(\rho'_1,\rho'_2,\cdots,\rho'_N,\beta)$.



(Subsec.2.1 is the $N=1$ case.) The Ricci tensor and the scalar curvature are, for N=2, given by [10]

$$
\begin{aligned}
&ds^2=dx^2+dy^2+\omega^2(x^2+y^2)d\tau^2\quad ,\\
(R_{AB})=\frac{1}{(r^2)^2}\left(\begin{array}{ccc} y^2 & -xy & 0\\ -yx & x^2 & 0\\ 0 & 0 & \omega^2(r^2)^2\end{array}\right)\quad ,\quad &R=\frac{2}{r^2}>0\quad ,\quad r^2=x^2+y^2\quad ,\\
\sqrt{G}=\omega\sqrt{x^2+y^2}\quad ,\quad &\sqrt{G}R=\frac{2\omega}{\sqrt{x^2+y^2}}
\end{aligned}
\qquad (27)
$$

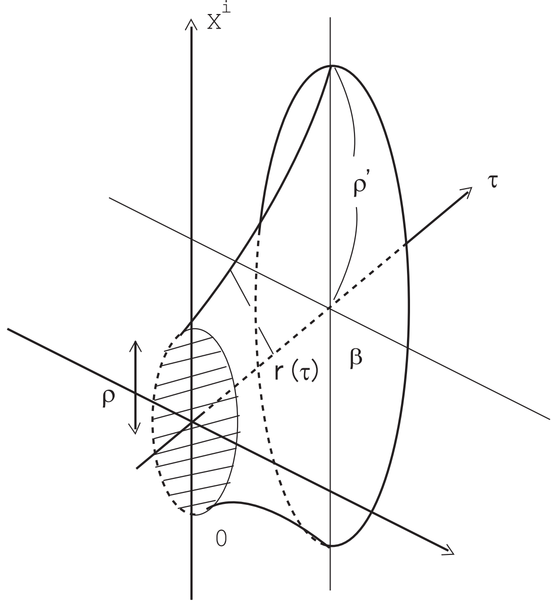
where $(X^1,X^2,X^3)=(x,y,\tau)$ is taken. (See eq.(63) in App.A, for the general N case using the general potential.)

We impose the periodicity (7)(period: β), and take a path $\{X^i=x^i(\tau)|\ 0\leq\tau\leq\beta,\ i=1,2,\cdots,N\}$(See Fig.4). The *induced* metric on the *line* is given by

$$
\begin{aligned}
X^i=x^i(\tau)\quad ,\quad dX^i=\dot{x}^i d\tau\quad ,\quad \dot{x}^i\equiv\frac{dx^i}{d\tau}\quad ,\quad 0\leq\tau\leq\beta\quad ,\\
i=1,2,\cdots,N\quad ,\quad ds^2=\sum_{i=1}^{N}((\dot{x}^i)^2+\omega^2(x^i)^2)d\tau^2\quad .
\end{aligned}
\qquad (28)
$$

[10] All curvature calculation in this work is checked by the algebraic calculation soft "Maxima"[26].

Figure 5: A path of hyper-surface. N(=2) dim hypersurface in N+1 dim space $(X^1, X^2, \cdots, X^N, \tau)$. S^{N-1} radius $r(\tau)$ starts with $r(0) = \rho$ and ends with $r(\beta) = \rho'$. We take this configuration as a path in the path integral (34) and (45). This is a closed-string configuration.



Then the *length* L of the path $\{x^i(\tau)\}$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\sum_{i=1}^N ((\dot{x}^i)^2 + \omega^2 (x^i)^2)}\, d\tau \quad . \tag{29}$$

We take the half of the length ($\frac{1}{2}L$) as the system Hamiltonian(*minimal length principle*). Then the free energy F of the system is given by

$$\mathrm{e}^{-\beta F} = (\prod_i \int_{-\infty}^{\infty} d\rho_i) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2} \int_0^\beta \sqrt{\sum_{i=1}^N ((\dot{x}^i)^2 + \omega^2 {x^i}^2)} d\tau \right] \quad ,(30)$$

where the path-integral is done for all possible paths $\{x^i(\tau); i = 1, 2, \cdots N\}$ with the indicated b.c.. We can regard this as the free energy for a variation ('Dirac' type) of the N harmonic oscillators's. (See next subsection for the ordinary type of the N harmonic oscillators.)

Instead of the length L, we can take another geometric quantity. Let us consider the following N dim *hypersurface* in N+1 dim space (a closed-string configuration). See Fig.5

for the N=2 case.

$$\sum_{i=1}^{N}(X^i)^2 = r^2(\tau) \quad , \quad \sum_{i=1}^{N} X^i dX^i = r\dot{r}d\tau \quad , \quad 0 \leq \tau \leq \beta \quad . \tag{31}$$

The form of $r(\tau)$ describes a path (N dimensional hypersurface in the bulk) which is *isotropic* in the 'brane' at τ (the N dim plane 'perpendicularly' standing at τ of the extra axis, not the hypersurface). The *induced* metric on the N dim hypersurface is given by

$$ds^2 = \sum_{i,j}(\delta_{ij} + \frac{\omega^2}{\dot{r}^2}x^i x^j)dx^i dx^j \equiv \sum_{i,j} g_{ij}dx^i dx^j \quad ,$$

$$g_{ij} = \delta_{ij} + \frac{\omega^2}{\dot{r}^2}x^i x^j \quad , \quad r^2 = \sum_{i=1}^{N}(x^i)^2 \quad , \quad \det(g_{ij}) = 1 + \frac{\omega^2 r^2}{\dot{r}^2} \quad . \tag{32}$$

This is the metric of a O(N) nonlinear system and is the one dimensional *nonlinear sigma model* as the field theory. [11] Then the *area* of the N dim hypersurface, A_N, is given by

$$A_N = \int \sqrt{\det g_{ij}}\; d^N x = \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2}\, r^{N-1} d\tau \quad . \tag{33}$$

When we take $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy F is given by

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2}\, r^{N-1} d\tau\right] \quad . \tag{34}$$

We should compare this result ($N = 4$) with the proposed 5D Casimir energy for the *flat* geometry (4). The component $\sqrt{\dot{r}^2 + \omega^2 r^2}$ in the integrand of (34) is replaced by $\sqrt{\dot{r}^2 + 1}$ in (4).

We recognize, if we start with

$$ds^2 = \sum_{i=1}^{N}(dX^i)^2 + d\tau^2 \quad \text{(N+1 dim Euclidean flat)} \quad , \tag{35}$$

instead of (26), the integration measure becomes *exactly* the same as (4). [12]

[11] The standard model (2 dim nonlinear sigma model) has often been used so far in order to show the *renormalization group* behavior of various systems. The background (effective action) formulation of the string theory heavily relies on the model.

[12] The starting line element (26) can be written as a general form: $ds^2 = \sum_{i=1}^{N}(dX^i)^2 + 2V(r)d\tau^2$. The content of this subsection is valid for this general potential $V(r)$. (35) is the case $V = 1/2$. See App.A.

4.2 Standard Type of N Harmonic Oscillators

Now we consider another type of N+1 dim Euclidean space (X^i,τ); $i=1,2,\cdots N$ described by the following line element.

$$
\begin{aligned}
ds^2 &= d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2+\omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2+2\omega^2\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\} \\
&= \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2+2V(r)d\tau^2\}^2 \quad , \quad V(r)=\frac{\omega^2}{2}r^2 \quad , \quad r^2=\sum_{i=1}^{N}(X^i)^2 \quad , \qquad (36)
\end{aligned}
$$

with the condition:

[Line Element Regularity Condition] :

$$
d\tau^2\sim O(\epsilon^2) \quad , \quad (dX^i)^2\sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2\}\sim O(1)\ , \qquad (37)
$$

in order to keep all terms of (36) in the order of ϵ^2. [13] Again we note that, in the above case, we do *not* have N+1 dim (bulk) metric. We impose the periodicity (7): (period: β).

Here we take a path of Fig.4: $\{x^i(\tau)|\ 0\leq\tau\leq\beta, i=1,2,\cdots,N\}$ and the *induced* metric on the path is given by

$$
\begin{aligned}
&X^i=x^i(\tau) \quad , \quad dX^i=\dot{x}^i d\tau \quad , \quad \dot{x}^i\equiv\frac{dx^i}{d\tau} \quad , \quad 0\leq\tau\leq\beta \quad , \\
&ds^2=[\sum_{i=1}^{N}((\dot{x}^i)^2+\omega^2(x^i)^2)]^2 d\tau^2 \quad . \qquad (38)
\end{aligned}
$$

Then the *length* L of the path $\{x^i(\tau)\}$ is given by

$$
L[x^i(\tau)]=\int ds=\int_0^\beta\sum_{i=1}^{N}((\dot{x}^i)^2+\omega^2(x^i)^2)d\tau \quad . \qquad (39)
$$

Hence, taking $\frac{1}{2}L$ as the Hamiltonian (*minimal length principle*), the free energy F of the system is given by

$$
\mathrm{e}^{-\beta F}=\left(\prod_i\int_{-\infty}^{\infty}d\rho_i\right)\int_{\substack{x^i(0)=\rho_i\\ x^i(\beta)=\rho_i}}\prod_{i,\tau}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\int_0^\beta\sum_{i=1}^{N}((\dot{x}^i)^2+\omega^2(x^i)^2)d\tau\right], (40)
$$

where the path-integral is done for all possible paths with the indicated b.c.. This is *exactly* the free energy of N harmonic oscillators.

We note again the condition (37) is necessary for the *elastic* view to the hyper-surface.

[13] As in (12), this condition restricts the trajectory configuration (38) only to *smooth* hyper-surfaces in the (N+1)-dim space.

4.3 Middle type of O(N) nonlinear system

Instead of (36), we can start from a slightly modified metric.

$$
\begin{aligned}
ds^2 &= \omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2 + 2\omega^2\kappa\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\} \\
&= \omega^2 r^2\left(\omega^2 r^2 d\tau^2 + 2\kappa\sum_{j=1}^{N}(dX^j)^2\right) = 4V(r)\left(V(r)d\tau^2 + \kappa\sum_{j=1}^{N}(dX^j)^2\right) \quad , \\
& V(r) = \frac{\omega^2}{2}r^2 \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad . \qquad (41)
\end{aligned}
$$

We drop the first term of (36), and add a free (real) parameter κ in the third one. We stress that, in this case, we need *not* the condition of (37). The line element is the ordinary one and we have the bulk metric G_{AB} in this case. The Ricci tensor and the scalar curvature, for $N = 2$, are given by

$$
\begin{aligned}
& ds^2 = \omega^4(x^2+y^2)^2 d\tau^2 + 2\omega^2\kappa(x^2+y^2)(dx^2+dy^2) \quad , \\
& (R_{AB}) = \frac{1}{(r^2)^2}\begin{pmatrix} 4y^2 & -4xy & 0 \\ -4yx & 4x^2 & 0 \\ 0 & 0 & \frac{2\omega^2}{\kappa}(r^2)^2 \end{pmatrix} \quad , \quad R = \frac{4}{\kappa\omega^2(r^2)^2} \quad , \quad r^2 = x^2+y^2 \quad , \\
& \sqrt{G} = 2\omega^4|\kappa|r^4 \quad , \quad \sqrt{G}R = 8\omega^2\cdot\mathrm{sign}(\kappa) \quad , \qquad (42)
\end{aligned}
$$

where $(X^1, X^2, X^3) = (x, y, \tau)$ and $\mathrm{sign}(\kappa)$ is the sign of κ. [14] (See eq.(75) in App.A for general potential.) We consider the N dim hypersurface (31), or Fig.5, and the *induced* metric on it is given by

$$
ds^2 = \sum_{i,j=1}^{N} 2\omega^2 r^2(\kappa\delta_{ij} + \frac{1}{2}\frac{\omega^2}{\dot{r}^2}x^i x^j)dx^i dx^j \equiv \sum_{i,j} g_{ij}dx^i dx^j \quad . \qquad (43)
$$

Then the *area* of this hypersurface is given by

$$
A_N = \int \sqrt{\det g_{ij}}\ d^N x = \frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta r^N\sqrt{\dot{r}^2 + \frac{r^2\omega^2}{2|\kappa|}}\ r^{N-1}d\tau \quad . \qquad (44)
$$

Taking $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy, F, is given by

$$
\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta r^N\sqrt{\dot{r}^2+\frac{r^2\omega^2}{2|\kappa|}}\ r^{N-1}d\tau\right] \quad .(45)
$$

We should compare this result (N=4,κ=1/2) with the proposed 5D Casimir energy for the *warped* geometry (4). They are similar ($(\omega r)^4\sqrt{\dot{r}^2 + r^2\omega^2}$ of (45) is replaced by $(1/\omega z)^4\sqrt{r'^2+1}$ of (4).). The exactly same one is obtained in the next subsection.

[14] $R > 0$ for $\kappa > 0$, $R < 0$ for $\kappa < 0$.

4.4 Modified type of O(N) nonlinear system

Instead of (41), we take the following modified type metric.

$$ds^2 = W(\tau)\left(2V(r)d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\right) \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad . \tag{46}$$

We recognize, if we start with $W(\tau) = \frac{1}{\tau^2},\ V(r) = \frac{1}{2}$:

$$\text{Euclidean (AdS)}_{N+1}: \quad ds^2 = \frac{1}{\tau^2}\{d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\} \quad , \tag{47}$$

instead of (41), the integration measure *exactly* becomes the same as the warped case in (4): $\tau^{-N}\sqrt{\dot{r}^2+1}r^{N-1}d\tau$.

The content in this section is generalized for the *general* isotropic potential in App.A.

5 Quantum Role of Space-Time Coordinates and the Matter Fields - New Treatment of Quantum Gravity -

The present standpoint on the metric field $G_{MN}(X)$ is that it is *not* the quantum-field variable and the form (the dependency on X) is *not* affected in the field-quantization process. All other fields (, other than the metric field, such as the electromagnetic fields, scalar fields, the fermion fields, the gluon fields, etc.) [15] are treated as the quantum-field variables. [16] The metric field plays only the role of the *background* or *fixed* field. The form of $G_{MN}(X)$ is, in principle, given by the solution of the field equation of the 'effective' action which is obtained after the *field*-quantization of all *matter* fields. The quantum behavior of the space-time is realized as the *statistical mechanics* of the coordinates X^M as described in the previous sections.

The *traditional* definition of Casimir Energy of the 5D electromagnetic field theory is, for the flat case (3),

$$\mathrm{e}^{-l^4 E_{Cas}} = \int \mathcal{D}A \exp\left[i\int d^4xdy(\mathcal{L}_{EM}^{5D} + \mathcal{L}_{gauge})\right]\Bigg|_{\text{Euclid}} \quad ,$$
$$\mathcal{L}_{EM}^{5D}[A_M(X)] = -\frac{1}{4}F_{MN}F^{MN} \ , \ F_{MN} = \partial_M A_N - \partial_N A_M \ , \ \mathcal{L}_{gauge}[A_M(X)] = -\frac{1}{2}(\partial_M A^M)^2 \ . \tag{48}$$

[15] We call them matter fields.

[16] The metric field is differently treated from the matter fields (all other fields) in the field-quantization process.

The expression of E_{Cas} defined above, is given by [22]

$$
\begin{aligned}
&\text{For Flat Geometry (5 dim elctromagnetism) :}\\
&E_{Cas}(l) = \int_{\tilde{p}\leq\Lambda}\frac{d^4p}{(2\pi)^4}\int_0^l dy(F_f^-(\tilde{p},y)+4F_f^+(\tilde{p},y)) \quad ,\\
&F_f^{\mp}(\tilde{p},y) = -\int_{\tilde{p}}^{\infty} d\tilde{k}\frac{\mp\cosh\tilde{k}(2y-l)+\cosh\tilde{k}l}{2\sinh(\tilde{k}l)} \quad .
\end{aligned}
\tag{49}
$$

The plus-minus symbol, $\mp$, indicates the contribution from Z_2-parity odd (-) and even (+) components. $\tilde{p}$ is the maginitude of 4D momentum $(p_a)=(p_1,p_2,p_3,p_4)$. The coincidence with the previous result[21] was confirmed[22]. As for the warped case (2), the *traditional* definition, for the 5D free scalar theory, is given by[23]

$$
\begin{aligned}
&\mathrm{e}^{-T^{-4}E_{Cas}} = \int\mathcal{D}\Phi\ \exp\left[i\int d^5X\sqrt{-G}\mathcal{L}_s^{5D}\right]\Bigg|_{\text{Euclid}}\\
&= \int\mathcal{D}\Phi(X)\exp\left[\left[\int d^4xdz\frac{1}{(\omega z)^5}\frac{1}{2}\Phi\{\omega^2z^2\partial_a\partial^a\Phi+(\omega z)^5\hat{L}_z\Phi\}\right]\right] \quad ,\\
&\mathcal{L}_s^{5D}[\Phi(X);X] = -\frac{1}{2}\nabla^M\Phi\nabla_M\Phi-\frac{1}{2}m^2\Phi^2 \quad ,\\
&\frac{1}{\omega}<|z|<\frac{1}{T} \quad , \quad \hat{L}_z=\frac{d}{dz}\frac{1}{(\omega z)^3}\frac{d}{dz}-\frac{m^2}{(\omega z)^5} \quad , \quad (m^2=-4\omega^2) \quad .
\end{aligned}
\tag{50}
$$

where $\hat{L}_z$ is the kinetic operator in the extra space (Bessel differential operator). Casimir energy E_{Cas} defined in (50) is explicitly given by

$$
\begin{aligned}
&\text{For Warped Geometry (5 dim Free Scalar, } m^2=-4\omega^2\text{):}\\
&-E_{Cas}^{\mp}(\omega,T) = \int\frac{d^4p_E}{(2\pi)^4}\bigg|_{\tilde{p}\leq\Lambda}\int_{1/\omega}^{1/T}dz\ F_w^{\mp}(\tilde{p},z) \quad , \quad F_w^{\mp}(\tilde{p},z)=\frac{1}{(\omega z)^3}\int_{\tilde{p}^2}^{\infty}\{G_k^{\mp}(z,z)\}dk^2 \quad ,\\
&G_p^{\mp}(z,z') = \mp\frac{\omega^3}{2}z^2z'^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z)\mp\mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z')\mp\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega})-\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})},\\
&(\hat{L}_z-p^2s(z))G_p^{\mp}(z,z') = \left\{\begin{array}{ll}\epsilon(z)\epsilon(z')\hat{\delta}(|z|-|z'|) & \text{for P}=-1\\ \hat{\delta}(|z|-|z'|) & \text{for P}=1\end{array}\right. \quad , \quad s(z)=\frac{1}{(\omega z)^3}
\end{aligned}
\tag{51}
$$

where $\mathbf{I}_0$ and $\mathbf{K}_0$ are the modified Bessel functions of 0-th order.

Casimir energy defined above, which has been traditionally calculated, gives Λ^5-divergence. The integral $\int\frac{d^4p_E}{(2\pi)^4}dz$ ($\int\frac{d^4p}{(2\pi)^4}dy$) appearing in eq.(51) ((49)) corresponds to the summation over all positions in 5 dim bulk space $\int d^4xdz$ ($\int d^4xdy$). The above expression says E_{Cas} is the total sum of $F(r^{-1},z)$ ($F(r^{-1},y)$) over the bulk space positions. We notice here the Λ^5 divergence comes from the fact that we have overlooked some proper *integration measure*. The summation, or the *averaging* procedure (of F) should be properly defined at this stage. In the present standpoint we regard the *coordinate* system (x^a,z) ((x^a,y)) as the *quantum*

statistical system and consider that the coordinate x^a is the *quantum mechanical* variable with the extra one z (y) as *Euclidean time*. The traditional treatment (simple summation over the set of positions) should be corrected by the present quantum (geometric) approach. We have proposed it should be done by the *path-integral* over all hypersurfaces in the bulk space (x^a, z) ((x^a, y)), as described in the previous sections. Hence the right expression of Casimir energy is given by (4).

6 Discussion and Conclusion

We have shown some quantum statistical systems of N variables can be described by the path (line or hypersurface) integral over the N+1 dim Euclidean space with an appropriate Hamiltonian (*length* of the line or *area* of the hypersurface). The system dynamics is determined by choosing the following two things: 1) With which bulk metric does one start and 2) which type of path (line or hypersurface) does one take. The choice 1) specifies the bulk geometry and the choice 2) specifies the *embedded geometry* of the path. This is the *geometric view* of the quantum statistical system. The result is applied to Casimir energy of 5 dim models and we show the proposed new definition (4) is valid.

As shown in (35) and (47), the bulk metrics for the integration measures (4) are standard. Hence the conditions (12) and (37) are *not* necessary only for the proof (of the correctness of (4)). But the conditions are important for the *elastic* (Harmonic oscillator) view of the hyper-surface (See (16) and (40)). More generally they are important when we view the hyper-surface as the *quantum mechanical system* of the potential $V(x)$ (See (25) and (73)). Hence, in this last paragraph, we argue the meaning of the line element regularity condition (12) or (37). Traditionally the quantum mechanics is formulated by using the operators $\{\hat{x}^i, \hat{p}^i | i = 1, 2, \cdots, N\}$ which satisfy *Heisenberg algebra*:

$$[\hat{x}^i\ , \hat{p}^j] = i\hbar\delta_{ij} \quad . \tag{52}$$

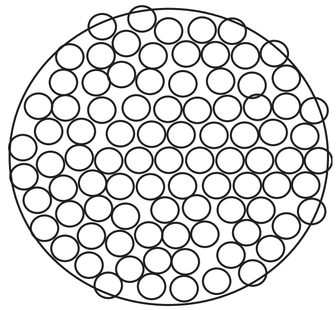
where $\hat{p}^i$ is the momentum operator conjugate of $\hat{x}^i$. This quantum system is characterized by the *uncertainty relation* among the expectation values of these operators :

$$\Delta x^i \Delta p^j \geq \frac{1}{2}\hbar\ \delta_{ij} \quad . \tag{53}$$

In relation to this uncertainty equation, let us discuss a possible meaning of the equations of (12) and (37). They guarantee the smooth surface (differentiable in the extra-coordinate direction). In this case, the metric does *not* exist in the bulk, instead we have the *induced metric* on the path (hypersurface). Although the geometric formulation is done in the N+1 dim bulk space, we can regard only the hypersurface as the 'ordinary' world in the sense that the metric is defined only on the hypersurface.

In the 'discrete' or 'regularized' level, we can take the following configuration as the 4D plane 'perpendicularly' standing at a fixed extra-axis (z or τ) point. We have the integral over 5D space, $\int d^4p_E \int dz$, in the expression of the AdS$_5$ Casimir energy (51). The 4D

Figure 6: Sphere lattice in the 4D Euclidean momentum space (p_1, p_2, p_3, p_4). The big 4D ball (radius Λ) is composed of many small 4D balls (radius μ). The density of small ones shows the 'resolution' of this regularization of the 4D continuous manifold.



integral part, $\int d^4 p_E$, can be regularized as

$$\begin{array}{r}\mu \leq \sqrt{{p_E}^2} \leq \Lambda \quad , \quad \omega^{-1} < z < T^{-1} \quad , \\ \mu : \text{ IR cutoff} \quad , \quad \Lambda : \text{ UV cutoff} \quad , \quad \omega : \text{ 5D bulk curvature} \quad , \\ T = \omega \mathrm{e}^{-l\omega} \quad (l : \text{ periodicity}) \quad ,\end{array} \tag{54}$$

where ${p_E}^2 \equiv {p_1}^2 + {p_2}^2 + {p_3}^2 + {p_4}^2$, $p_4 = i\ p_0$. The finite range of z comes from the fact taht the AdS$_5$ geometry (bulk curvature ω) is the concerned manifold and we take the periodic b.c. (periodicity l) w.r.t. the extra axis. The restricted range, $\mu \leq \sqrt{{p_E}^2} \leq \Lambda$, for the Euclidean 4D momentum (p^a) comes from the regularization of this continuous 4D manifold ('Brane'). We can express this regularization as the sphere-lattice shown in Fig.6.

As the regularization, usually μ and Λ are taken independent as far as the following condition:

$$\mu \quad \ll \quad \Lambda \quad , \tag{55}$$

is satisfied. In the present treatment, however, we take a special regularization by imposing the equality between $\frac{\Lambda}{\mu}$ and $\frac{\omega}{T}$.[27]

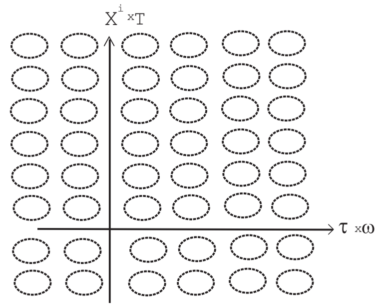
$$\text{IR-UV harmonic relation} \ : \quad \frac{\Lambda}{\mu} = \frac{\omega}{T} \gg 1 \quad . \tag{56}$$

We call this equality condition "IR-UV harmonic relation". This relation helps us to regularize both 4D manifold and the extra world in the 'harmonious' way. This is a simple way to reduce the number of independent regularization parameters.

With the momentum cut-off parameter Λ for the UV-regularization, and μ for the IR-regularization, the condition, (12) or (37), can be rewritten as

$$\frac{1}{d\tau^2} dX^2 < \frac{\omega^2}{T^2} = \frac{\Lambda^2}{\mu^2} \sim \infty \quad \text{or} \quad \frac{1}{\sqrt{d\tau^2}} \sqrt{dX^2} < \frac{\omega}{T} = \frac{\Lambda}{\mu} \sim \infty \quad , \tag{57}$$

Figure 7: Eqs.(58) and (59) suggest the foam-like structure in the bulk space (X^i, τ). Each "cell" has the size $\alpha_\tau \times \alpha_X$ approximately in the length units shown in the figure.



where the 'IR-UV harmonic relation' $\mu = \Lambda T/\omega$ is used. [17] The above relation looks like a sort of the *uncertainty* relation. It is the relation between 4D coordinates and the extra one , not between coordinates and momenta which makes the phase space. [18] We need the constraint, expressed by (57), on the bulk space coordinates when we regard the hypersurface as the elastic system. If we write the relation (57) as

$$dX^2 < \mathcal{H}^2 d\tau^2 \quad \text{or} \quad \sqrt{dX^2} < \mathcal{H}\sqrt{d\tau^2} \quad , \quad \mathcal{H} \equiv \omega/T \quad , \tag{58}$$

it says the length relation between the "smallest" interval in the X-direction and that in the τ-direction. Here we define the *dimensionless constant* $\mathcal{H}$ by the ratio of ω and T. If we can take the assumption:

$$\sqrt{T^2 dX^2} \sim \alpha_X \quad , \quad \sqrt{\omega^2 d\tau^2} \sim \alpha_\tau \quad , \quad \alpha_X < \alpha_\tau \quad , \tag{59}$$

where α_X and α_τ are regarded dimensionless constants of the order $O(1)$, it suggests the foam-like structure[29] of 5D bulk space. See Fig.7. The size of the one 'bubble' is about $T^{-1} \times \omega^{-1}$. Mathematically it suggests a new algebra among the bulk space coordinates.

7 Appendix A : General Quantum Statistical System of N Coordinates

We consider the system of N coordinates, $\{x^1, x^2, \cdots x^N\}$, in the general isotropic potential. This is the generalization of Sec.4 where the elastic-type potential is only considered.

[17] The relation appears in the regularization process of the numerical evaluation of (4)[23]. The choice was taken in Ref.[27] for the β-function calculation of the 5D warped YM theory. They regarded the parameters ω and T as "physical" UV and IR cutoffs respectively.

[18] In the development of the string theory, the uncertainty relation between the coordinates is shown to appear[28].

7.1 'Dirac' Type

Let us consider N+1 dim Euclidean space $(X^i,\tau), i=1,2,\cdots,N$ described by the following metric.

$$
\begin{aligned}
ds^2=\sum_{i=1}^{N}(dX^i)^2+2V(r)d\tau^2=G_{AB}dX^AdX^B \quad ,\\
A,B=1,2,\cdots,N,N+1;\quad X^{N+1}\equiv\tau \quad ,\\
(G_{AB})=\mathrm{diag}(1,1,\cdots,1,2V(r)) \quad ,\quad r^2\equiv\sum_{i=1}^{N}(X^i)^2 \quad ,\quad (X^A)=(X^i,\tau) \quad ,
\end{aligned}
\tag{60}
$$

where the isotropy property in N dim space $\{X^i|i=1\sim N\}$ is assumed. The general potential V depends only on r . The present convention is given by

$$
\begin{aligned}
\Gamma^A_{BC}=\frac{1}{2}G^{AD}(\partial_BG_{DC}+\partial_CG_{DB}-\partial_DG_{BC}) \quad ,\\
R^C_{\ \ D,AB}=\partial_A\Gamma^C_{BD}+\Gamma^C_{EA}\Gamma^E_{DB}-A\leftrightarrow B \quad ,\\
R_{AB}(=R_{BA})=R^C_{\ \ A,BC}=\partial_B\Gamma^C_{CA}-\partial_C\Gamma^C_{AB}+\Gamma^C_{DB}\Gamma^D_{AC}-\Gamma^C_{DC}\Gamma^D_{AB} \quad ,\quad R=G^{AB}R_{AB} \quad .
\end{aligned}
\tag{61}
$$

(Sec.3 is the $N=1$ case. Sec.4 is the case of the potential: $V(r)=\omega^2r^2/2$) The explicit result for (60) is

$$
\begin{aligned}
R_{ij}=\frac{V'}{2rV}\delta_{ij}+\{\frac{V''}{2V}-\frac{V'}{2rV}-\frac{1}{4}(\frac{V'}{V})^2\}\frac{X^iX^j}{r^2} \quad ,\\
R_{\tau i}=0 \quad ,\quad R_{i\tau}=0 \quad ,\quad R_{\tau\tau}=V''-\frac{V'^2}{2V}+(N-1)\frac{1}{r}V' \quad ,\\
R=\frac{V''}{V}-\frac{1}{2}(\frac{V'}{V})^2+(N-1)\frac{V'}{rV} \quad ,\quad \sqrt{G}=\sqrt{2V} \quad ,
\end{aligned}
\tag{62}
$$

where $V'=\frac{dV}{dr}, V''=\frac{d^2V}{dr^2}$. The elastic system (of N 'particles') is obtained by setting $V=\omega^2r^2/2$.

$$
\begin{aligned}
R_{ij}=\frac{\delta_{ij}}{r^2}-\frac{X^iX^j}{(r^2)^2} \quad ,\quad R_{\tau i}=0 \quad ,\quad R_{i\tau}=0 \quad ,\quad R_{\tau\tau}=(N-1)\omega^2 \quad ,\\
R=\frac{2(N-1)}{r^2} \quad ,\quad \sqrt{G}R=2(N-1)\frac{\omega}{r} \quad .
\end{aligned}
\tag{63}
$$

We impose the periodicity (7)(period: β). Here we take a path $\{X^i=x^i(\tau)|\ 0\leq\tau\leq\beta,\ i=1,2,\cdots,N\}$ and the *induced* metric on the line is given by

$$
\begin{aligned}
X^i=x^i(\tau) \quad ,\quad dX^i=\dot{x}^id\tau \quad ,\quad \dot{x}^i\equiv\frac{dx^i}{d\tau} \quad ,\quad 0\leq\tau\leq\beta \quad ,\\
ds^2=\left(\sum_{i=1}^{N}(\dot{x}^i)^2+2V(r)\right)d\tau^2 \quad .
\end{aligned}
\tag{64}
$$

See Fig.4. Then the length L of the path $\{x^i(\tau)\}$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\left(\sum_{i=1}^{N} (\dot{x}^i)^2 + 2V(r) \right)}\, d\tau \quad . \tag{65}$$

Taking the half of the length ($\frac{1}{2}L$) as the system Hamiltonian(*minimal length principle*), the free energy F of the system is given by

$$\mathrm{e}^{-\beta F} = (\prod_i \int_{-\infty}^{\infty} d\rho_i) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2} \int_0^\beta \sqrt{\left(\sum_{i=1}^{N} (\dot{x}^i)^2 + 2V(r) \right)}\, d\tau \right] \quad ,(66)$$

where the path-integral is done for all possible paths $\{x^i(\tau); i = 1, 2, \cdots N\}$ with the indicated b.c.. We can regard this as the free energy for the general system of N coordinates isotropically interacting.

Instead of the length L, we take another geometric quantity. Let us consider the N dim *hypersurface* in N+1 dim space (a closed-string configuration), (31). See Fig.5 for the N=2 case. The form of $r(\tau)$ describes a path (N dimensional hypersurface in the bulk) which is *isotropic* in the 'brane' at τ (the N dim plane 'perpendicularly' standing at τ of the extra axis, not the hypersurface). The *induced* metric on the N dim hypersurface is given by

$$ds^2 = \sum_{i,j} (\delta_{ij} + \frac{2V(r)}{r^2 \dot{r}^2} x^i x^j) dx^i dx^j \equiv \sum_{i,j} g_{ij} dx^i dx^j \quad ,$$

$$g_{ij} = \delta_{ij} + \frac{2V(r)}{r^2 \dot{r}^2} x^i x^j \quad , \quad r^2 = \sum_{i=1}^{N} (x^i)^2 \quad , \quad \det(g_{ij}) = 1 + \frac{2V(r)}{\dot{r}^2} \quad . \tag{67}$$

This is the metric of a O(N) nonlinear system. Then the area of the N dim hypersurface is given by

$$A_N = \int \sqrt{\det g_{ij}}\, d^N x = \frac{N \pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + 2V(r)} r^{N-1} d\tau \quad . \tag{68}$$

When we take $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy F is given by

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2} \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + 2V(r)}\, r^{N-1} d\tau \right] \quad .(69)$$

7.2 Standard Type

Now we consider another type of N+1 dim Euclidean space $(X^i,\tau);\ i=1,2,\cdots N$ described by the following line element.

$$
\begin{aligned}
ds^2 &= d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2 + 4V(r)^2 d\tau^2 + 4V(r)\{\sum_{j=1}^{N}(dX^j)^2\} \\
&= \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2 + 2V(r)d\tau^2\}^2 \quad , \quad r=\sqrt{\sum_{i=1}^{N}(X^i)^2} \quad ,
\end{aligned} \tag{70}
$$

with the condition (37) in order to keep all terms of (70) in the order of ϵ^2. Again we note that, in the above case, we do *not* have N+1 dim (bulk) metric. We impose the periodicity (7): (period: β).

Here we take a path $\{x^i(\tau)|\ 0\leq\tau\leq\beta, i=1,2,\cdots,N\}$(Fig.4) and the *induced* metric on the path is given by

$$
\begin{aligned}
&X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0\leq\tau\leq\beta \quad , \\
&ds^2 = [\sum_{i=1}^{N}(\dot{x}^i)^2 + 2V(r)]^2 d\tau^2 \quad , \quad r=\sqrt{\sum_{i=1}^{N}(x^i)^2} \quad .
\end{aligned} \tag{71}
$$

Then the length L of the path $\{x^i(\tau)\}$ is given by

$$
L[x^i(\tau)] = \int ds = \int_0^\beta \left(\sum_{i=1}^{N}(\dot{x}^i)^2 + 2V(r)\right) d\tau \quad . \tag{72}
$$

Hence, taking $\frac{1}{2}L$ as the Hamiltonian (*minimal length principle*), the free energy F of the system is given by

$$
\mathrm{e}^{-\beta F} = \left(\prod_i \int_{-\infty}^{\infty} d\rho_i\right) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{i,\tau}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\int_0^\beta\left(\sum_{i=1}^{N}(\dot{x}^i)^2 + 2V(r)\right)d\tau\right] \quad ,(73)
$$

where the path-integral is done for all possible paths with the indicated b.c.. This is the free energy for the general isotropic system of N coordinates.

7.3 Middle type of O(N) nonlinear system

Instead of (70), we can start from a slightly modified metric.

$$
ds^2 = 4V(r)^2 d\tau^2 + 4\kappa V(r)\{\sum_{j=1}^{N}(dX^j)^2\} = 4V(r)\left\{V(r)d\tau^2 + \kappa\sum_{j=1}^{N}(dX^j)^2\right\} \quad . \tag{74}
$$

We drop the first term of (70), and add a free (real) parameter κ in the third one. We stress that, in this case, we need *not* the condition of (37). The line element is the ordinary type and we have the bulk metric G_{AB} in this case. The Ricci tensor and the scalar curvature are given by

$$
\begin{aligned}
&(G_{AB}) = \mathrm{diag}(4\kappa V,\ 4\kappa V,\ \cdots,\ 4\kappa V,\ 4V^2) \quad , \quad \sqrt{G} = \sqrt{\det G_{AB}} = (4|\kappa|V)^{N/2}\cdot 2V \quad , \\
&R_{ij} = \left\{\frac{N}{2}\frac{1}{r}(\frac{V'}{rV})' - \frac{N-2}{4}(\frac{V'}{rV})^2\right\}X^iX^j + \left\{\frac{1}{2}(\frac{V'}{rV})'r + N\frac{V'}{rV} + \frac{N}{4}r^2(\frac{V'}{rV})^2\right\}\delta_{ij} \quad , \\
&R_{\tau\ i} = R_{i\ \tau} = 0 \quad , \quad R_{\tau\tau} = \frac{1}{\kappa}\left\{(\frac{V'}{r})'r + N\frac{V'}{r} + \frac{N-2}{2}\frac{V'^2}{V}\right\} \quad , \quad V' = \frac{d}{dr}V(r) \quad , \\
&R = \frac{1}{4\kappa}\left\{(N+1)\frac{r}{V^2}(\frac{V'}{r})' + \frac{N^2-3N-2}{4}\frac{V'^2}{V^3} + N(N+1)\frac{V'}{rV^2}\right\} \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad ,(75)
\end{aligned}
$$

where $i,j = 1,2,\cdots N$ and $X^{N+1} = \tau$. [19] We consider the N dim hypersurface (31), or Fig.5, and the *induced* metric on it is given by

$$
\begin{aligned}
ds^2 = \sum_{i,j=1}^{N} 4V(r)(\kappa\delta_{ij} + \frac{V(r)}{r^2\dot{r}^2}x^ix^j)dx^idx^j \equiv \sum_{i,j} g_{ij}dx^idx^j \quad , \\
g_{ij} = 4V(r)(\kappa\delta_{ij} + \frac{V(r)}{r^2\dot{r}^2}x^ix^j) \quad . \qquad (76)
\end{aligned}
$$

Then the area of this hypersurface is given by

$$
A_N = \int \sqrt{\det g_{ij}}\ d^Nx = \frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta V^{N/2}\sqrt{\dot{r}^2 + \frac{V(r)}{|\kappa|}}\ r^{N-1}d\tau \quad . \qquad (77)
$$

Taking $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy is given by

$$
\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta V^{N/2}\sqrt{\dot{r}^2 + \frac{V(r)}{|\kappa|}}\ r^{N-1}d\tau\right] \quad .(78)
$$

7.4 Modified type

The general modified metric is given by

$$
ds^2 = W(\tau)\left\{2V(r)d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\right\} \quad , \qquad (79)
$$

where $V(r)$ and $W(\tau)$ are general functions of r and τ respectively. (As pointed out in Sec.4.4, the special case: $W(\tau) = \frac{1}{\tau^2}$, $V(r) = \frac{1}{2}$ is Euclidean AdS_{N+1}.)

[19] $R > 0$ for $\kappa > 0$, $R < 0$ for $\kappa < 0$.

The Ricci tensor and the scalar curvature are given by

$$
\begin{aligned}
(G_{AB}) = \mathrm{diag}(W(\tau),\ W(\tau),\ \cdots,\ W(\tau),\ 2W(\tau)V(r)) \quad ,\\
\sqrt{G} = \sqrt{\det G_{AB}} = \sqrt{2}W(\tau)^{(N+1)/2}V(r)^{1/2} \quad ,\\
R_{ij} = \left\{\frac{1}{2}\frac{1}{rV}(\frac{V'}{r})' - \frac{1}{4}(\frac{V'}{rV})^2\right\} X^i X^j + \left\{\frac{1}{4V}\partial_\tau(\frac{\dot{W}}{W}) + \frac{1}{2}\frac{V'}{rV} + \frac{N-1}{8}(\frac{\dot{W}}{W})^2\frac{1}{V}\right\}\delta_{ij} \quad ,\\
R_{\tau\ i} = R_{i\ \tau} = -\frac{N-1}{4}\frac{V'}{V}\frac{\dot{W}}{W}\frac{X^i}{r} \quad , \quad R_{\tau\tau} = \frac{N}{2}\partial_\tau(\frac{\dot{W}}{W}) + (\frac{V'}{r})'r + N\frac{V'}{r} - \frac{V'^2}{2V} \quad ,\\
V' = \frac{d}{dr}V(r) \quad , \quad \dot{W} = \frac{dW}{d\tau} \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad ,\\
R = \frac{1}{W}\left\{\frac{r}{V}(\frac{V'}{r})' - \frac{1}{2}(\frac{V'}{V})^2 + N\frac{V'}{rV} + \frac{N}{2}\frac{1}{V}\frac{\ddot{W}}{W} + \frac{N(N-5)}{8}\frac{1}{V}(\frac{\dot{W}}{W})^2\right\} \quad . \qquad (80)
\end{aligned}
$$

We consider the N dim hypersurface (31), or Fig.5, and the *induced* metric on it is given by

$$
\begin{aligned}
ds^2 = W(\tau)\sum_{i,j=1}^{N}\left(\delta_{ij} + \frac{2V(r)}{r^2\dot{r}^2}x^i x^j\right)dx^i dx^j \equiv \sum_{i,j} g_{ij}dx^i dx^j \quad ,\\
g_{ij} = W(\tau)\left(\delta_{ij} + \frac{2V(r)}{r^2\dot{r}^2}x^i x^j\right) \quad . \qquad (81)
\end{aligned}
$$

Then the area of this hypersurface is given by

$$
A_N = \int \sqrt{\det g_{ij}}\ d^N x = \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta W(\tau)^{N/2}\sqrt{\dot{r}^2 + 2V(r)}\ r^{N-1}d\tau \quad . \qquad (82)
$$

Taking $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy is given by

$$
\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta W(\tau)^{N/2}\sqrt{\dot{r}^2 + 2V(r)}\ r^{N-1}d\tau\right] \quad . \qquad (83)
$$

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